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On quasi-minimal structures

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1 Introduction

Unlike the model thory of $(\mathbf{C}, +, \cdot, 0, 1)$, we do not know hardly anything about the model thoery of $(\mathbf{C}, +, \cdot, \exp, 0, 1)$. This situation is very different from the one concerning the model theory of $(\mathbf{R}, +, \cdot, <, \exp, 0, 1)$ or of $(\mathbf{R}, +, \cdot, <, 0, 1, f)_{f\in An([0,1])}$, where $An([0,1]) = \{f \mid f : U \to \mathbf{R}$ is analytic for U some open $\supset [0,1]^n\}$.

First attempts to investigate the model theory of $(\mathbf{C}, +, \cdot, \exp, 0, 1)$ are made by B. Zil'ber who has conjectured that the structure is a quasi-minimal structure which is a generalization of minimal structures.

Definition 1. An uncountable structue is called *quasi-minimal* if its definable sets are at most countable or co-countable.

The conjecture has not yet been answered neither affirmatively nor negatively. As a minor contribution to this line of research we study basic properties of quasi-minimal structures. It is well known that we can define a combinatrial geometry on minimal structures using a closure operation. It is then very natual to define a similar geometry on quasi-minimal structures.

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2 Pre-Geometry

In this note we only work with countable languages L. We also assume that the reader is familiar with basic model theory.

Definition 2. Let M be an uncountable structure and $A \subset M$. Then

$$\mathrm{ccl}_M(A) = \{b \in M : b \models \varphi, \varphi^M \text{ is countable for some } \varphi \in L(A)\}$$

We omit the subscript M if it is clear from context.

Definition 3. Let X be a set and cl be a function from $P(X)$ to $P(X)$, where $P(X)$ denotes the set of all subsets of X. If X and the function cl satisfy the following properties, we say that (X, cl) is a *pre-geometry*. Let $A \subset X$ and $b, c \in X$.

(I) $A \subset \mathrm{cl}(A)$.

(II) (Finite Character) $b \in \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$.

(III) (Transfer Property) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

(IV) (Exchange Property) $b \in \mathrm{cl}(Ac) - \mathrm{cl}(A) \Rightarrow c \in \mathrm{cl}(Ab)$.

Let M be an uncountable structure. We first show that (M, ccl) satisfies these properties under some conditions.

Proposition 4. For any infinite structure M, (M, ccl) satisfies (I) and (II).

Proof: Clear by the definition of ccl since the language is countable.

Lemma 5. Suppose M is a quasi-minimal structure. Let $A \subset M$, $|A| < |M|$ and $b, c \in M - \mathrm{ccl}(A)$. Then $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$.

Proof: If $\mathrm{tp}(b/A) \neq \mathrm{tp}(c/A)$, then there is a formula $\varphi(x) \in L(A)$ such that both $\varphi(b)$ and $\neg\varphi(c)$ hold. By the quasi-minimality of M, either φ or $\neg\varphi$ is countable. Hence either b or c is in $ccl(A)$. This contradicts to the assumption on b, c.

Proposition 6. Assume that M is quasi-minimal and homogeneous. Then (M, ccl) satisfies the transfer property (III).

Proof: Let $A \subset M$. We show that $\mathrm{ccl}(\mathrm{ccl}(A)) = \mathrm{ccl}(A)$. Clearly $\mathrm{ccl}(\mathrm{ccl}(A)) \supset \mathrm{ccl}(A)$ holds by (I). For the other direction, it is enough to show that $\mathrm{ccl}(\mathrm{ccl}(A)) \subset \mathrm{ccl}(A)$ for finite $A \subset M$, since $\mathrm{ccl}(A) = \cup\{\mathrm{ccl}(B) : B \subset M, |B| < \aleph_0\}$ by (II). Assume that there is an element $b \in \mathrm{ccl}(\mathrm{ccl}(A)) - \mathrm{ccl}(A)$. Since $|\mathrm{ccl}(A)| \leq |A| + |L|$, there is an element $c \in M - \mathrm{ccl}(\mathrm{ccl}(A))$. By Lemma 5, we have $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$. So by the homogeneity assumption on M, there is an A-automorphism f of M such that $f(b) = c$. Since b is in $\mathrm{ccl}(\mathrm{ccl}(A))$, c is also in $\mathrm{ccl}(\mathrm{ccl}(A))$. This contradicts to the assumption on c.

Proposition 7. Assume that M is quasi-minimal, homogeneous and $|M| \geq \aleph_2$. Then (M, ccl) satisfies the exchange property (IV).

Proof: By the finite character (II), it is enough to show the exchange property (IV) assuming that $A \subset M$ is finite. Suppose that there are elements $b, c \in M$ such that $b \in \mathrm{ccl}(Ac) - \mathrm{ccl}(A)$ and $c \notin \mathrm{ccl}(Ab)$. By Lemma 5, we have $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$. Let $p(x, y) = \mathrm{tp}(bc/A)$. We construct a sequence $(b_i)_{i \leq \omega_1}$ such that $b_0 = b$, $b_1 = c$ and $i < j \Rightarrow \mathrm{tp}(b_i b_j/A) = \mathrm{tp}(bc/A)$. Suppose that we have chosen $b_j (j < i)$.

Claim. $\cap_{j<i} p(b_j, M) \neq \emptyset$.

Proof of claim: Since $\cap_{j<i} p(b_j, M) = M - \cup_{j<i}(M - p(b_j, M))$, it is enough to show that $M - p(b_j, M)$ is countable for each j. Let $d \in M - p(b_j, M)$. Then there is a formula $\varphi(b_j, y) \in p(b_j, y)$ such that $\neg\varphi(b_j, d)$ holds. Since $\varphi(b, c)$ and $\mathrm{tp}(b/A) = \mathrm{tp}(b_j/A)$, $\varphi(b_j, y)$ is not countable in M by the homogeneity of M. Hence $\neg\varphi(b_j, y)$ is countable by the quasi-minimality of M and $d \in \mathrm{ccl}(Ab_j)$. So $M - p(b_j, M)$ is countable. This completes the proof of Claim.

Now we finish the proof of proposition. Let $b_i \in \cap_{j<i} p(b_j, M)$. By the definition of b_i, $B = \{b_i : i < \omega_1\} \subset \mathrm{ccl}(Ab_{\omega_1})$. But B is uncountable. This is a contradiction.

3 Some Examples

1. Any strongly minimal structure is quasi-minimal.

2. Let M be uncountable, P a unary predicate and $|M^P|$ coutable, then (M, P) is quasi-minimal.

3. Let T be a theory of an equivalence relation E with infinitely many infinite equivalent classes. Then T has quasi-minimal models such as;

   (a) E is an equivalence relation with uncountably many countable equivalence classes.

   (b) E is an equivalence relation with one uncountable class and countable countable classes.

4. There are no quasi-minimal random graphs.

   Proof: Assume that (M, R) is a quasi-minimal random graph. Let $a \in M$ and $b, c \in M - \mathrm{ccl}(a)$. By Lemma 5, we may assume without loss of generality that any element in $M - \mathrm{ccl}(a)$ is connected to a. Since M is a random graph, there is an element $d \in M$ such that $R(d, a) \wedge R(d, b) \wedge \neg R(d, c)$ holds. Then $d \in \mathrm{ccl}(a)$, because $R(d, a)$ holds. Hence $\mathrm{tp}(b/d) = \mathrm{tp}(c/d)$, but $R(d, b) \wedge \neg R(d, c)$ holds as well. This is a contradiction.

5. (a) $(\omega_1, <)$ is not quasi-minimal, since the successor points are definable.

   (b) $(\omega_1 \times \mathbf{Z}, <)$ ($<$ is the lexicographic order) is quasi-minimal but not homogeneous.

   (c) $(\omega_1 \times \mathbf{Q}, <)$ is quasi-minimal and homogeneous. But the exchange property (IV) does not holds (since the cardinality is less than $\aleph_2$).

4 Remarks

Unlike strongly minial sets, the first-order property of quasi-minimal sets are not easily understood. The reason for this is that for two elementarily equivalent structures M and N, the quasi-minimality of M may or may not imply the quasi-minimality of N. This forbids us to employ the usual model theoretic tools such as compactness arguments.

Another difficulty is that the luck of natural interesting examples. Zil'ber's conjecture on the structute $(\mathbf{C}, +, \cdot, \exp, 0, 1)$ seems plausible but at this moment we do not know how to study the structure. As a very small first step we notice the following:

Remark 8. It seems very natural to claim that for a quasi-minimal structure $(M, \cdots)$, the expanded structure $(M, \cdots, P_i(i \in \omega))$ is also quasi-minimal where each P_i is a unary predicate whose interpretation is a countable subset of M. As a corollary to this we have that $(\mathbf{C}, +, \cdot, 0, 1, P_i(i \in \omega))$ is quasi-minimal where each P_i is a unary predicate whose interpretation is a countable subset of $\mathbf{C}$.

Remark 9. In Section 2 we studied the basic pre-geometric properties of quasi-minimal sets. Although our proof used the additional homogeneity assumption on the structure, it is not clear whether this assumption is necessary.

Remark 10. In model theory we often work in a saturated model. It seems that our usual arguments for constructing saturated models are not enough to define a saturated quasi-minimal structures.

5 References

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